1. The Method of Dominant Balance

The method of dominant balance can be used to determine the order of magnitude of terms in an equation. Let us examine a low order polynomial first to acquant ourselves with the method.

Solve

$$x^3 + x^2 - \epsilon = 0 \tag{1.1}$$

The core of a dominant balance is assuming that two terms of our equation "balance" as $\epsilon \to 0^+$ and then showing that the remaining terms are smaller as $\epsilon \to 0^+$. By balanced we mean that two terms are of the same order of magnitude and we use the notation $x \sim \mathcal{O}(\epsilon)$ to denote the fact that $x = C\epsilon$ where $C = \mathcal{O}(1)$.

First we see that we cannot balance x^2 with x^3, so we must either balance ϵ with either x^2 or x^3. Let us first assume that ϵ balances with x^3. We write

$$x^3 \sim \mathcal{O}(\epsilon) \tag{1.2a}$$

$$\implies x \sim \mathcal{O}(\epsilon^{1/3}) \tag{1.2b}$$

The term left over is $x^2 \sim \mathcal{O}(\epsilon^{2/3})$. But $\epsilon^{2/3} >> \epsilon$ as $\epsilon \to 0^+$. This is inconsistent, we want the terms not in our dominant balance to be smaller! So we are now left with the only other possibility, that

$$x^2 \sim \mathcal{O}(\epsilon) \tag{1.3a}$$

$$\implies x \sim \mathcal{O}(\epsilon^{1/2}) \tag{1.3b}$$

Now the term left over is $x^3 \sim \mathcal{O}(\epsilon^{3/2})$, which satisfies $\epsilon^{3/2} << \epsilon^{1/2}$ (or equivalently $x^3 << x$ as $x \to 0^+$).

We now see that the roots of (1.1) which vanish as $\epsilon \to 0^+$ satisfy

$$x \sim \mathcal{O}(\epsilon^{1/2}) \tag{1.4}$$

This tells us the proper asymptotic expansion to use, which is:

$$x = \sum_{k=0} \epsilon^{k/2} x_k = x_0 + \epsilon^{1/2} x_1 + \epsilon x_2 + \cdots \tag{1.5}$$