GRADUATE SEMINAR SPRING 2007 PROBLEM SET 1

JONATHAN LETO

1. Problem 1

Show that for $a_i > 0, p_i > 0$

$$\lim_{n\to\infty} \sqrt[n]{p_1 a_1^n + p_2 a_2^n + \cdots p_l a_l^n} = \max a_i \tag{1.1}$$

First, suppose that a_1 and a_2 are the maximum elements of a_i. That means that $a_1 = a_2$ and therefore we can write $p_1 a_1^n + p_2 a_2^n = \widetilde{p_1} a_1^n$ where $\widetilde{p_1} = p_1 + p_2$, and we now have a set of $l-1$ distinct constants which have a unique maximum. Obviously, if the maximum is shared by k constants then we can redefine our p_i's such that we have a set of $l-k$ distinct constants. We may therefore consider the case where a_i has a unique maximum. Let $\max a_i = a_k$ with $1 \le k \le l$.

We now wish to pull out the largest term inside the square root, a_k. For a fixed n we have

(1.2)

$$a_k \sqrt[n]{p_1} \le a_k \sqrt[n]{p_1 \left(\frac{a_1}{a_k}\right)^n + p_2 \left(\frac{a_2}{a_k}\right)^n + \cdots + p_k + \cdots + p_l \left(\frac{a_l}{a_k}\right)^n} \le a_k \sqrt[n]{p_1 + p_2 + \cdots p_l}$$

Using the fact that $\lim_{n\to\infty} a^{1/n} = a^0 = 1$ for $a > 0$, we can take the limit of the inequality as $n \to \infty$, which gives us an upper and lower bound of a_k. Thus, by the Squeezing Theorem, we have that

$$\lim_{n\to\infty} \sqrt[n]{p_1 a_1^n + p_2 a_2^2 + \cdots p_l a_l^n} = a_k = \max a_i \tag{1.3}$$

□

2. Problem 2

Show that for $a_i > 0, p_i > 0$

$$\lim_{n\to\infty} \frac{p_1 a_1^{n+1} + p_2 a_2^{n+1} + \cdots p_l a_l^{n+1}}{p_1 a_1^n + p_2 a_2^n + \cdots p_l a_l^n} = \max a_i \tag{2.1}$$

If there is more than one maximum of a_i we may proceed as in the previous problem by redefining the p_i's, so it is sufficient to consider only the case of a a_i having a unique maximum. Let $\max a_i = a_k$ with $1 \le k \le l$.

Similar to the previous problem, we will pull out the largest factors of the numerator and denominator, a_k. For a fixed n we have

$$s_n = \frac{a_k^{n+1}\left(p_1\left(\frac{a_1}{a_k}\right)^{n+1} + p_2\left(\frac{a_2}{a_k}\right)^{n+1} + \cdots + p_k + \cdots + p_l\left(\frac{a_l}{a_k}\right)^{n+1}\right)}{a_k^n\left(p_1\left(\frac{a_1}{a_k}\right)^{n} + p_2\left(\frac{a_2}{a_k}\right)^{n} + \cdots + p_k + \cdots + p_l\left(\frac{a_l}{a_k}\right)^{n}\right)} \tag{2.2}$$

Since $\frac{a_i}{a_k} < 1$ for all $i \neq k$, in taking the limit we find

$$\lim_{n\to\infty} s_n = \frac{a_k^{n+1}p_k}{a_k^n p_k} = a_k = \max a_i \tag{2.3}$$

□

3. Problem 3

Let $f(x)$ be a polynomial whose zeros are all real and positive and for which

$$-\frac{f'(x)}{f(x)} = c_0 + c_1x + c_2x^2 + \cdots + c_kx^k + \cdots \tag{3.1}$$

Show that

$$\lim_{k\to\infty} \frac{1}{\sqrt[k]{c_k}} = \lim_{k\to\infty} \frac{c_{k-1}}{c_k} \tag{3.2}$$

exists and that it is equal to the smallest zero of $f(x)$.

Let $f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$. We recognize the left hand side of (3.1) as the logarithmic derivative, so we have

$$-\frac{f'(x)}{f(x)} = -\frac{\mathrm{d}}{\mathrm{dx}}\left(\log f(x)\right) \tag{3.3a}$$

$$= -\frac{\mathrm{d}}{\mathrm{dx}}\left(\sum_{i=1}^{n} \log\left(x-\alpha_i\right)\right) \tag{3.3b}$$

$$= -\sum_{i=1}^{n} \frac{1}{x-\alpha_i} \tag{3.3c}$$

$$= \sum_{i=1}^{n} \frac{1}{\alpha_i - x} \tag{3.3d}$$

$$= \sum_{i=1}^{n} \frac{1}{\alpha_i(1-\frac{x}{\alpha_i})} \tag{3.3e}$$

$$= \sum_{i=1}^{n} \frac{1}{\alpha_i}\left(1 + \frac{x}{a_i} + \left(\frac{x}{a_i}\right)^2 + \cdots\right) \tag{3.3f}$$

$$= \sum_{i=1}^{n} \frac{1}{\alpha_i} + x\sum_{i=1}^{n}\left(\frac{1}{\alpha_i}\right)^2 + \cdots + x^k\sum_{i=1}^{n}\left(\frac{1}{\alpha_i}\right)^{k+1} \tag{3.3g}$$

We have now found the constants c_k which are given by

$$c_k = \sum_{i=1}^{n}\left(\frac{1}{\alpha_i}\right)^{k+1} \tag{3.4}$$

Let α_j denote the smallest of the α_i. Since all of the roots α_i are positive, all of the terms in the sum are positive, so we easily find a lower bound

$$\sqrt[k]{\left(\frac{1}{a_j}\right)^{k+1}} = \frac{1}{\alpha_j}\sqrt[k]{\frac{1}{\alpha_j}} \leq \sqrt[k]{c_k} \tag{3.5}$$

Since α_j is the smallest root, the largest term in the sum is $\left(\frac{1}{\alpha_j}\right)^{k+1}$. We can then find an upper bound as

$$\sqrt[k]{c_k} \leq \sqrt[k]{n\left(\frac{1}{\alpha_j}\right)^{k+1}} = \frac{1}{\alpha_j}\sqrt[k]{\frac{n}{\alpha_j}} \tag{3.6}$$

Putting these results together we have

$$\frac{\alpha_j}{\sqrt[k]{\frac{1}{\alpha_j}}} \leq \frac{1}{\sqrt[k]{c_k}} \leq \frac{\alpha_j}{\sqrt[k]{\frac{n}{\alpha_j}}} \tag{3.7}$$

In the limit as $k \to \infty$ the denominators converge to 1 and therefore by the Squeeze theorem $\lim_{n\to\infty} \frac{1}{\sqrt[k]{c_k}} = \alpha_j$, the smallest root of $f(x)$. As for the showing that $\lim_{n\to\infty} \frac{c_{k-1}}{c_k} = \alpha_j$ one may use exactly the same method as in Problem 2, which is also a ratio of finite polynomials. □

4. Problem 4

Let $f(x)$ and $\phi(x)$ be positive, continuous functions on $[a, b]$. Show that

$$\lim_{n\to\infty} \sqrt[n]{\int_a^b \phi(x)\,(f(x))^n\,dx} \tag{4.1}$$

exists and is equal to the maximum of $f(x)$ on $[a, b]$.

Let us assume that $f(x)$ attains a single maximum at $x = \zeta$ in the interval $[a, b]$, denoted by $\max_{[a,b]} f(x) = f(\zeta)$. We can now write the integral with the maximum value of the integrand "factored out":

$$f(\zeta)\sqrt[n]{\int_a^b \phi(x)\left(\frac{f(x)}{f(\zeta)}\right)^n dx} \tag{4.2}$$

If we can bound this from above and below by sequences that converge to $f(\zeta)$, then the Squeeze Theorem can be used. Since $\left(\frac{f(x)}{f(\zeta)}\right)^n < 1$ we can easily find an upper bound that converges to $f(\zeta)$

$$f(\zeta)\sqrt[n]{\int_a^b \phi(x)\left(\frac{f(x)}{f(\zeta)}\right)^n dx} \leq f(\zeta)\sqrt[n]{\int_a^b \phi(x)dx} \tag{4.3}$$

To get a lower bound takes a bit more work. Because $f(x)$ is continuous we know that for any $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that

$$f(\zeta) - \epsilon \leq f(x) \leq f(x) + \epsilon \tag{4.4}$$

for $\zeta - \delta \leq x \leq \zeta + \delta$. Using this we can find a lower bound

$$f(\zeta) \sqrt[n]{\int_{\zeta-\delta}^{\zeta+\delta} \phi(x) \left(\frac{f(\zeta)-\epsilon}{f(\zeta)}\right)^n dx} \leq f(\zeta) \sqrt[n]{\int_a^b \phi(x) \left(\frac{f(x)}{f(\zeta)}\right)^n dx} \tag{4.5}$$

Since everything inside the integrand is constant except $\phi(x)$ we have

$$f(\zeta) \sqrt[n]{\int_{\zeta-\delta}^{\zeta+\delta} \phi(x) \left(\frac{f(\zeta)-\epsilon}{f(\zeta)}\right)^n dx} = (f(\zeta)-\epsilon) \sqrt[n]{\int_{\zeta-\delta}^{\zeta+\delta} \phi(x) dx} \tag{4.6}$$

Now $\sqrt[n]{\int_{\zeta-\delta}^{\zeta+\delta} \phi(x)dx}$ is just a positive number which converges to 1 as $n \to \infty$. We now have

$$f(\zeta) - \epsilon \leq \lim_{n\to\infty} f(\zeta) \sqrt[n]{\int_a^b \phi(x) \left(\frac{f(x)}{f(\zeta)}\right)^n dx} \leq f(\zeta) \tag{4.7}$$

Now let $\epsilon \to 0$ and we have the required result. □